Yangians and Finite $W$-algebras

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1. Introduction

- A finite $W$-algebra is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a complex semi-simple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element.

- A finite $W$-algebra is a generalization of the universal enveloping algebra $U(\mathfrak{g})$. For $e = 0$ it coincides with $U(\mathfrak{g})$.

- Finite $W$-algebra is a quantization of the Poisson algebra of functions on the Slodowy (i.e. transversal) slice at $e$ to the orbit $Ad(G)e$, where $\mathfrak{g} = Lie(G)$.

- Finite $W$-algebras for semi-simple Lie algebras were introduced by A. Premet.

Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over $\mathbb{C}$.

**Definition.** An element $e \in \mathfrak{g}$ is **nilpotent**, if $\text{ad}(e)$ is a nilpotent endomorphism of $\mathfrak{g}$.

**Definition.** A nilpotent element $e \in \mathfrak{g}$ is **regular** nilpotent, if its centralizer $\mathfrak{g}^e = \text{Ker} \; \text{ad}(e)$ has dimension as small as possible. This minimal dimension equals the **rank** $\mathfrak{g}$.
Example. $g = \mathfrak{sl}(n)$.

$e \in \mathfrak{sl}(n)$ is nilpotent if and only if $e$ is an $n \times n$-matrix with eigenvalues zero.

$e$ is a **regular** nilpotent $\iff$ its Jordan normal form contains a single Jordan block

\[
e = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Theorem. (Jacobson-Morozov)
Associated to a nonzero nilpotent element \( e \in \mathfrak{g} \),
there always exists an \( \mathfrak{sl}(2) \)-triple \( \{ e, h, f \} \) which satisfies
\[
[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.
\]

Proof. Induction on \( \dim \mathfrak{g} \).
2. \(\mathbb{Z}\)-GRADINGS

Definition. A Dynkin \(\mathbb{Z}\)-grading.

Let \(\mathfrak{sl}(2) = \langle e, h, f \rangle\). The eigenspace decomposition of the adjoint action

\[ \text{ad}(h) : \mathfrak{g} \rightarrow \mathfrak{g} \]

provides a \(\mathbb{Z}\)-grading:

\[ \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \quad \mathfrak{g}_j = \{ x \in \mathfrak{g} \mid \text{ad}(h)(x) = jx \}. \]

Properties:

1. \( e \in \mathfrak{g}_2 \),
2. \( \text{ad}(e) : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j+2} \) is injective for \( j \leq -1 \),
3. \( \text{ad}(e) : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j+2} \) is surjective for \( j \geq -1 \).
Definition. A good $\mathbb{Z}$-grading.

A $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ for a semi-simple $\mathfrak{g}$ is called a good $\mathbb{Z}$-grading for $e$, if it satisfies the conditions (1)-(3).

For a reductive $\mathfrak{g}$, there is an additional condition: the center of $\mathfrak{g}$ is in $\mathfrak{g}_0$.

Remark.

Dynkin $\mathbb{Z}$-grading $\implies$ Good $\mathbb{Z}$-grading

Dynkin $\mathbb{Z}$-grading $\nRightarrow$ Good $\mathbb{Z}$-grading
3. Definition of finite $W$-algebras

$\mathfrak{g}$ is a reductive Lie algebra,

$(\cdot | \cdot)$ is a non-degenerate invariant symmetric bilinear form,

$e$ is a nilpotent element,

$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is a good $\mathbb{Z}$-grading for $e$,

$\chi \in \mathfrak{g}^*$  $\chi(x) := (x | e) \ \forall x \in \mathfrak{g}$.

Define a bilinear form on $\mathfrak{g}_{-1}$ as

$$(x, y) := ([x, y] | e) = \chi([x, y]) \ \forall x, y \in \mathfrak{g}_{-1}$$

**Remark.** The bilinear form on $\mathfrak{g}_{-1}$ is skew-symmetric and non-degenerate.

**Proof.** The skew-symmetry follows by definition.

The non-degeneracy follows from the bijection

$ad(e) : \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_1$

and the identity

$$(x, y) = (x | [y, e]).$$

Hence $\dim \mathfrak{g}_{-1}$ is even.
Pick a Lagrangian (a maximal isotropic) subspace $l$ of $g_{-1}$ with respect to the form $(\cdot, \cdot)$. Then $\dim l = \frac{1}{2} \dim g_{-1}$.

Let $m = (\oplus_{j \leq -2} g_j) \oplus l$.

The restriction of $\chi$ to $m$
\[ \chi : m \rightarrow \mathbb{C} \]
defines a one-dimensional representation $\mathbb{C}_\chi$ of $m$ thanks to the Lagrangian condition on $l$.

Let $I_\chi$ be the left ideal of $U(g)$ generated by $a - \chi(a)$ for $a \in m$.

**Definition.** The generalized Whittaker module is

\[ Q_\chi := U(g) \otimes_{U(m)} \mathbb{C}_\chi \cong U(g)/I_\chi. \]

**Definition.** The finite $W$-algebra associated to the nilpotent element $e$ is

\[ W_\chi := \text{End}_{U(g)}(Q_\chi)^{op}. \]
Remark. $W_\chi$ can be identified as the space of Whittaker vectors in $U(g)/I_\chi$.

Let $\pi : U(g) \to U(g)/I_\chi$ be the natural projection, and let $y \in U(g)$.

$$W_\chi = (Q_\chi)^{adm} = \{ \pi(y) \in U(g)/I_\chi | [a, y] \in I_\chi \quad \forall a \in m \}.$$ 

The multiplication is

$$\pi(y_1)\pi(y_2) = \pi(y_1y_2)$$

for $y_i \in U(g)$ such that $[a, y_i] \in I_\chi \quad \forall a \in m$ and $i = 1, 2$.

Remark. The isoclasses of finite $W$-algebras do not depend on good $\mathbb{Z}$-grading and Lagrangian subspace $l$.

Example. Let $e = 0$. Then $\chi = 0$, $g_0 = g$, $m = 0$,

$$Q_\chi = U(g), \quad W_\chi = U(g).$$


If $g$ is a reductive Lie algebra and $e \in g$ is a regular nilpotent element, then $W_\chi \cong Z(g)$, the center of $U(g)$. 
**Definition.** *Kazhdan filtration* on $W_\chi$.

Define the $\mathbb{Z}$-grading on $T(\mathfrak{g})$ induced by the shift by 2 of the fixed good $\mathbb{Z}$-grading. For $X \in \mathfrak{g}_j$ set

$$\text{deg } X = j + 2.$$  

This induces a filtration on $U(\mathfrak{g})$, and therefore on $U(\mathfrak{g})/I_\chi$ and on $W_\chi \subset U(\mathfrak{g})/I_\chi$.


Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then the associated graded algebra $Gr(W_\chi)$ is isomorphic to $S(\mathfrak{g}^e)$.
4. Pyramids

Let $\mathfrak{g} = \mathfrak{gl}(n)$.

- The classification of good $\mathbb{Z}$-gradings for $e \in \mathfrak{gl}(n)$ is described in terms of *pyramids*. They are the tool to encode the information needed to define a finite $W$-algebra.

**Definition.** For a given tuple $(q_1, q_2, \ldots, q_l)$ of positive integers, a diagram $\pi$ consists of $q_i$ boxes stacked in column $i$ for $1 \leq i \leq l$.

$\pi$ is called a *pyramid*, if each row is a connected horizontal strip.

**Example.** $\mathfrak{g} = \mathfrak{gl}(8)$, $n = 8$.

```
 1
 2   5   7
 3   4   6   8
```

is not a pyramid.
is a pyramid: \( \pi = (2, 2, 3, 1), \ l = 4. \)

**The height of** \( \pi \) **is** \( m = \max q_i. \)

**The nilpotent** \( e \) **associated to** \( \pi \) **is**

\[
e := \sum_{i,j} e_{i,j}
\]

**Example.** If \( \pi = (2, 2, 3, 1) \), then the height is \( m = 3 \),

\[
e = e_{1,3} + e_{3,6} + e_{2,4} + e_{4,7} + e_{7,8}
\]

**The Jordan block sizes** of the matrix \( e \) **are precisely the lengths of the rows** in \( \pi \):

\[
(p_1, p_2, \ldots, p_m), \text{ where } 0 \leq p_1 \leq p_2 \leq \ldots \leq p_m = l
\]

If \( \pi = (2, 2, 3, 1) \), then the Jordan type of the matrix \( e \) is \((1, 3, 4)\).
• From \( \pi \) we also read off a certain **shift matrix** \( \sigma = (s_{i,j}) \) of size \( m \times m \).

\( s_{i,j} \) is \# of bricks in the \( i \)-th row indented from the \( j \)-th row
at the *left* edge of the pyramid, if \( i \geq j \),

at the *right* edge of the pyramid, if \( i \leq j \).

**Example.** If \( \pi = (2, 2, 3, 1) \)

\[
\sigma = \begin{pmatrix}
0 & 0 & 1 \\
2 & 0 & 1 \\
2 & 0 & 0
\end{pmatrix}
\]
**Remark.** *Special case: σ is the zero matrix*

\[ \sigma = (0) \]

\[ \iff \text{all rows have the same length: } p_1 = p_2 = \ldots = p_m = l \]

\[ \iff \text{all Jordan blocks of the matrix } e \text{ are of the same size } l. \]

The size of \( \sigma \) is \( \frac{n}{l} = \# \) of Jordan blocks in the matrix \( e \).
Pyramid $\pi$ defines $\mathbb{Z}$-grading of $\mathfrak{gl}(n)$, which is good for $e$:

\[ \mathfrak{g} = \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}(r) \quad \text{deg}(e_{i,j}) := \text{col}(j) - \text{col}(i), \quad \mathfrak{h} := \mathfrak{g}(0) \]

**Example.**

$\mathfrak{g} = \mathfrak{gl}(3), \ n = 3$

\[
\begin{pmatrix}
0 & 1 & 1 \\
-1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\]

$\mathbb{Z}$-grading on the elementary matrices:

\[
\begin{pmatrix}
0 & 1 & 1 \\
-1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\]

**Remark.** Note that $e \in \mathfrak{g}(1)$.

We double the degree to agree with the previous definition.

**Definition.** $W(\pi)$ is the finite $W$-algebra associated to the pyramid $\pi$. 
5. **Finite W-algebras and Yangians**

- For a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$, the **Yangian** of $\mathfrak{g}$ is an infinite-dimensional *Hopf algebra* $Y(\mathfrak{g})$. It is a deformation of the universal enveloping algebra of the Lie algebra of polynomial currents of $\mathfrak{g}$: $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$.

**Definition.** The Yangian $Y_n = Y(\mathfrak{gl}(n))$ is an associative unital algebra over $\mathbb{C}$ with a countable set of generators

$$T_{i,j}^{(r)} \text{ where } 1 \leq i, j \leq n, \text{ and } r \geq 0.$$ 

We employ the formal series in $Y_n[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{i,j} \cdot 1 + T_{i,j}^{(1)}u^{-1} + T_{i,j}^{(2)}u^{-2} + \ldots$$

- Relations in $Y_n$:

$$(u - v)[T_{i,j}(u), T_{k,l}(v)] = T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u).$$
• The **evaluation homomorphism** \( \text{ev} : Y_n \rightarrow U(\mathfrak{gl}(n)) \) is defined by

\[
\text{ev}(T_{i,j}^{(r)}) = \begin{cases} 
  e_{i,j} & \text{if } r = 1, \\
  0 & \text{if } r > 1
\end{cases}
\]

• \( Y_n \) is a **Hopf algebra** with **comultiplication** given by

\[
\Delta(T_{i,j}^{(r)}) = \sum_{s=0}^{r} \sum_{k} T_{i,k}^{(s)} \otimes T_{k,j}^{(r-s)}.
\]

The map

\[
\Delta_l : Y_n \rightarrow Y_n^\otimes l,
\]

where

\[
\Delta_l := \Delta_{l-1,l} \circ \cdots \circ \Delta_{2,3} \circ \Delta
\]

is a homomorphism of associative algebras.
**Definition.** The **truncated Yangian** $Y^l_n$ of level $l$ is the image of $Y_n$ under the homomorphism

$$
\text{ev}^\otimes l \circ \Delta_l : Y_n \longrightarrow [U(\mathfrak{gl}(n))]^\otimes l.
$$


Let $\pi$ be a pyramid for $\mathfrak{gl}(n)$. Assume that the nilpotent $e$ associated to $\pi$ consists of $\frac{n}{l}$ Jordan blocks each of the same size $l$. Then

$$W(\pi) \cong Y^l_n \subset [U(\mathfrak{gl}(n))]^\otimes l \cong U(\mathfrak{h})$$
J. Brundan and A. Kleshchev generalized this result to an arbitrary nilpotent $e \in \mathfrak{gl}(n)$ and obtained a realization of the finite $W$-algebra for the general linear Lie algebra as a quotient of a so-called shifted Yangian, which is associated to the matrix $\sigma$ (Adv. Math. 2004). They proved that

$$W(\pi) \cong Y_m^l(\sigma),$$

where $m$ is the height of the pyramid $\pi$, $l$ is length of the bottom row, and $\sigma$ is the shift matrix.

**Remark.**

*Shifted truncated Yangian* $Y_m^l(\sigma)$ is a quotient of subalgebra $Y_m(\sigma) \subset Y_m$, determined by the matrix $\sigma$. 
Example.

(1) $\mathfrak{g} = \mathfrak{gl}(3)$, $n = 3, l = 3$

$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ \\ 2 & 4 & 1 \end{pmatrix}$

$e = e_{1,2} + e_{2,3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is regular nilpotent, $W(\pi) \cong Y_1^3$

(2) $\mathfrak{g} = \mathfrak{gl}(4)$, $n = 4, l = 2$

$\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$e = e_{1,3} + e_{2,4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $W(\pi) \cong Y_2^2$
(3) $\mathfrak{g} = \mathfrak{gl}(3)$, $n = 3$, $l = 2$

$$\sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e = e_{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W(\pi) \cong Y_2^2(\sigma) - \text{shifted Yangian}$$
6. Finite $W$-algebras for Lie superalgebras

Let $\mathfrak{g}$ be a classical simple Lie superalgebra, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\mathfrak{g}_0$ is a reductive Lie algebra, and $\mathfrak{g}$ has a $\mathfrak{g}$-invariant super-symmetric bilinear form.

Let $e \in \mathfrak{g}_0$ be an even nilpotent element, and we fix $\mathfrak{sl}(2) = \langle e, h, f \rangle$ (even).

- The above definition of $W_\chi$ makes sense, however Theorem of Kostant does not hold in this case since $W_\chi$ must have a non-trivial odd part, and the center of $U(\mathfrak{g})$ is even.

- Kazhdan filtration on $W_\chi$ can be defined exactly as in the Lie algebra case.

- $Gr(W_\chi)$ is supercommutative.
Conjecture

Assume that $\mathfrak{g}$ is a Lie superalgebra with reductive even part $\mathfrak{g}_0$.

If $\dim(\mathfrak{g}_{-1})$ is even, then $Gr_K W_\chi \simeq S(\mathfrak{g}^e)$

If $\dim(\mathfrak{g}_{-1})$ is odd, then $Gr_K W_\chi \simeq S(\mathfrak{g}^e) \otimes \mathbb{C}[\xi]$, where $\mathbb{C}[\xi]$ is the exterior algebra generated by one element $\xi$.

• We proved this Conjecture in the case when $\mathfrak{g} = Q(n)$ or $D(2, 1; \alpha)$ and $e$ is regular nilpotent.

• Y. Zheng and B. Shu proved this Conjecture for a basic Lie superalgebra $\mathfrak{g}$ over $\mathbb{C}$ of any type except $D(2, 1; \alpha)$, where $\alpha \not\in \bar{\mathbb{Q}}$ (J. Algebra, 2015)
• J. Brown, J. Brundan and S. Goodwin proved that the finite $W$-algebra for $\mathfrak{g} = \mathfrak{gl}(m|n)$ associated to regular (principal) nilpotent element is a certain truncation of a shifted version of the super-Yangian $Y(\mathfrak{gl}(1|1))$ (*Algebra Number Theory*, 2013)
7. The super-Yangian of $\mathfrak{gl}(1|1)$

$$\mathfrak{gl}(1|1) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}\} \quad [A, B] = AB - (-1)^{p(A)p(B)}BA$$

**Definition.** The super-Yangian $Y_{1|1} = Y(\mathfrak{gl}(1|1))$ is an associative unital superalgebra over $\mathbb{C}$ with a countable set of generators

$$T_{i,j}^{(r)}$$ where $i, j = 1, 2$, and $r \geq 0$.

The $\mathbb{Z}_2$-grading of $Y_{1|1}$ is defined by

$$p(T_{i,j}^{(r)}) = p(i) + p(j).$$

We employ the formal series:

$$T_{i,j}(u) = \sum_{r \geq 0} T_{i,j}^{(r)} u^{-r} \in Y_{1|1}[[u^{-1}]].$$

• Relations in $Y_{1|1}$:

$$(u - v)[T_{i,j}(u), T_{k,l}(v)] = (-1)^{p(i)p(k) + p(i)p(l) + p(k)p(l)}((T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)).$$
• Gauss factorization:

\[
T(u) := \begin{pmatrix} T_{1,1}(u) & T_{1,2}(u) \\ T_{2,1}(u) & T_{2,2}(u) \end{pmatrix} = F(u)D(u)E(u)
\]

\[
D(u) = \begin{pmatrix} d_1(u) & 0 \\ 0 & d_2(u) \end{pmatrix}, \quad E(u) = \begin{pmatrix} 1 & e(u) \\ 0 & 1 \end{pmatrix}, \quad F(u) = \begin{pmatrix} 1 & 0 \\ f(u) & 1 \end{pmatrix}
\]

\[
d_i(u) = \sum_{r \geq 0} d_i^{(r)} u^{-r}, \quad e(u) = \sum_{r \geq 1} e^{(r)} u^{-r}, \quad f(u) = \sum_{r \geq 1} f^{(r)} u^{-r}
\]

• Drinfeld generators: \( Y_{1|1} \) is generated by even elements \( d_1^{(r)}, d_2^{(r)} \) for \( r > 0 \), and odd elements \( e^{(r)}, f^{(r)} \) for \( r > 0 \).
8. Shifted super-Yangian $Y_{1|1}(\sigma)$

Let

$$\sigma = \begin{pmatrix} 0 & s_{1,2} \\ s_{2,1} & 0 \end{pmatrix}, \text{ where } s_{1,2}, s_{2,1} \geq 0 \text{ are integers}$$

**Definition.** $Y_{1|1}(\sigma)$ is a subalgebra of $Y_{1|1}$ generated by $d_1^{(r)}, d_2^{(r)}$ for $r > 0$, $e^{(r)}$ for $r > s_{1,2}$ and $f^{(r)}$ for $r > s_{2,1}$.

- If $\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $Y_{1|1}(\sigma) = Y_{1|1}$. 
9. **Principal $W$-algebra $W(\pi)$**

$\mathfrak{g} = \mathfrak{gl}(l|k)$ is a general linear Lie superalgebra. Assume that $l \geq k$.

**Definition.** $\pi$ is a two-rowed **pyramid:**

$k$ is the number of boxes in the 1-st row,

$l$ is the number of boxes in the 2-nd row.

**Definition.** The **shift matrix** for $\pi$ is

$$\sigma = \begin{pmatrix} 0 & s_{1,2} \\ s_{2,1} & 0 \end{pmatrix},$$

where $\pi$ has

$s_{2,1}$ columns of height one on its left side and

$s_{1,2}$ columns of height one on its right side,

or if $k = 0$ and $l = s_{2,1} + s_{1,2}$.

- $l = s_{2,1} + k + s_{1,2}$. 
Example.

\[ g = \mathfrak{gl}(5|2), \quad \pi = \begin{array}{cccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 \end{array} \quad \sigma = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \]

\[ g = \mathfrak{gl}(2|2), \quad \pi = \begin{array}{cc} 3 & 4 \\ 1 & 2 \end{array} \quad \sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

- Pyramid \( \pi \) defines \( \mathbb{Z} \)-grading on \( g \):

\[ g = \bigoplus_{r \in \mathbb{Z}} g(r) \quad \text{deg}(e_{i,j}) := \text{col}(j) - \text{col}(i), \quad h := g(0) \]

- The explicit \textbf{principal} (regular) nilpotent element \( e \) is

\[ e := \sum_{i,j} e_{i,j} \in g_0 \]

summing over all \textit{adjacent pairs} of boxes in \( \pi \).

- The principal \( W \)-algebra \( W(\pi) \) associated to the pyramid \( \pi \) is defined as usual.
Theorem. (Brown-Brundan-Goodwin, 2012)
Assume that $e$ is a principal (regular) nilpotent element.

Special Case: $g = \mathfrak{gl}(l|l)$, $\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$W(\pi) \cong Y_{1|1}^l,$$

that is the image of $Y_{1|1}$ under the homomorphism

$$\text{ev}^\otimes l \circ \Delta_l : Y_{1|1} \rightarrow [U(\mathfrak{gl}(1|1))]^\otimes l.$$

The General Case: $g = \mathfrak{gl}(l|k)$.

$$W(\pi) \cong Y_{1|1}^l(\sigma) \subset U(\mathfrak{gl}_1)^{\otimes s_{2,1}} \otimes U(\mathfrak{gl}(1|1))^{\otimes k} \otimes U(\mathfrak{gl}_1)^{\otimes s_{1,2}} \cong U(\mathfrak{h})$$

where $U(\mathfrak{gl}_1) := \mathbb{C}[e_{1,1}]$, $l = s_{2,1} + k + s_{1,2}$.

$$Y_{1|1}^l(\sigma) \cong Y_{1|1}(\sigma)/I^l(\sigma),$$

where $I^l(\sigma)$ is the two-sided ideal generated by $d_{1}^{(r)}$ for $r > k$. 
Peng described the finite $W$-algebra for $\mathfrak{g} = \mathfrak{gl}(M|N)$ associated to an $e$ in the case when the Jordan type of $e$ satisfies the following condition:

$$e = e_M \oplus e_N,$$

where $e_M$ is principal nilpotent in $\mathfrak{gl}(M|0)$ and the sizes of the Jordan blocks of $e_N$ are all greater or equal to $M$.

Signed pyramids: $M$ is the number of boxes with $+$

$N$ is the number of boxes with $-$

The top row of $\pi$ is the only row assigned with $+$
Example.

\[ g = \mathfrak{gl}(2|7), \quad \pi = \begin{array}{cccc}
1 & 2 \\
2 & 4 & 6 \\
1 & 3 & 5 & 7
\end{array} \quad l = 4, \quad \sigma = \begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} \]

Peng proved that \( W(\pi) \cong Y^l_{1|n}(\sigma) \), where \( n + 1 \) is the height of the pyramid \( \pi \), \( l \) is the length of the bottom row, and \( \sigma \) is the shift matrix.

- The general case when even nilpotent \( e \in \mathfrak{gl}(M|N) \) could be arbitrary, is highly challenging!
10. The queer Lie superalgebra \( \mathfrak{g} = \mathcal{Q}(n) \)

\[
\mathcal{Q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \text{ are } n \times n \text{ matrices} \right\}
\]

\(e_{i,j}\) and \(f_{i,j}\) are standard bases in \(A\) and \(B\) respectively:

\[
e_{i,j} = \begin{pmatrix} E_{ij} & 0 \\ 0 & E_{ij} \end{pmatrix}, \quad f_{i,j} = \begin{pmatrix} 0 & E_{ij} \\ E_{ij} & 0 \end{pmatrix}
\]

\(z = \sum_{i=1}^{n} e_{i,i}\) is a central element

\(\mathcal{Q}(n)\) admits an odd nondegenerate \(\mathfrak{g}\)-invariant super-symmetric bilinear form

\[(x|y) := otr(xy) \text{ for } x, y \in \mathfrak{g},\]

where \(otr \begin{pmatrix} A & B \\ B & A \end{pmatrix} = trB.\)


Let $\mathfrak{sl}(2) = \langle e, h, f \rangle$, where

$$e = \sum_{i=1}^{n-1} e_{i,i+1}, \quad h = \text{diag}(n - 1, n - 3, \ldots, 3 - n, 1 - n), \quad f = \sum_{i=1}^{n-1} i(n - i)e_{i+1,i}.$$ 

$e$ is a **regular** nilpotent element.

$h$ defines an **even** Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$ whose degrees on the elementary matrices are

$$
\begin{pmatrix}
0 & 2 & \cdots & 2n - 2 & 0 & 2 & \cdots & 2n - 2 \\
-2 & 0 & \cdots & 2n - 4 & -2 & 0 & \cdots & 2n - 4 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
2 - 2n & \cdots & 0 & 2 - 2n & \cdots & 0 \\
0 & 2 & \cdots & 2n - 2 & 0 & 2 & \cdots & 2n - 2 \\
-2 & 0 & \cdots & 2n - 4 & -2 & 0 & \cdots & 2n - 4 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
2 - 2n & \cdots & 0 & 2 - 2n & \cdots & 0
\end{pmatrix}
$$
Replace \( e = \sum_{i=1}^{n-1} e_{i,i+1} \) by \( E = \sum_{i=1}^{n-1} f_{i,i+1} \Rightarrow E \) is odd.

Define \( \chi \in g_0^* \) by \( \chi(x) = (x|E) \).

\[ m := \bigoplus_{j=2}^{n} g_{2-2j}. \]

It is generated by \( e_{i+1,i} \) and \( f_{i+1,i} \) where \( i = 1, \ldots, n-1 \)

\[ \chi(e_{i+1,i}) = 1, \quad \chi(f_{i+1,i}) = 0 \]

• The left ideal \( I_\chi \) and \( W_\chi \) are defined now as usual.

\[ Gr(W_\chi) \cong S(g^E), \quad \dim(g^E) = (n|n) \]
12. Generators of $W_\chi$

- A. Sergeev defined by induction the elements $e_{i,j}^{(m)}$ and $f_{i,j}^{(m)}$ belonging to $U(Q(n))$:

\[
e_{i,j}^{(m)} = \sum_{k=1}^{n} e_{i,k} e_{k,j}^{(m-1)} + (-1)^{m+1} \sum f_{i,k} f_{k,j}^{(m-1)},
\]

\[
f_{i,j}^{(m)} = \sum_{k=1}^{n} e_{i,k} f_{k,j}^{(m-1)} + (-1)^{m+1} \sum f_{i,k} e_{k,j}^{(m-1)},
\]

\[
e_{i,j}^{(0)} = \delta_{i,j}, \quad f_{i,j}^{(0)} = 0.
\]

**Theorem.** (P–S)

$W_\chi$ has $n$ even generators: $\pi(e_{n,1}^{(n+k-1)})$ and $n$ odd generators: $\pi(f_{n,1}^{(n+k-1)})$, $k = 1, \ldots, n$. 
13. The Harish-Chandra homomorphism

\[ \mathfrak{p} := \bigoplus_{j=0}^{n-1} \mathfrak{g}_{2j} \]

is a parabolic subalgebra of \( \mathfrak{g} \).

\[ \mathfrak{h} := \mathfrak{g}_0 = \langle e_{i,i} \mid f_{i,i} \rangle \]

is a Cartan subalgebra of \( \mathfrak{g} \).

- Since the good \( \mathbb{Z} \)-grading is even, then \( W_\chi \) can be regarded as a subalgebra of \( U(\mathfrak{p}) \).

\[ U(\mathfrak{p})^+ := \bigoplus_{i>0} U(\mathfrak{p})_i \]

is a two sided ideal in \( U(\mathfrak{p}) \) and \( U(\mathfrak{p})/U(\mathfrak{p})^+ \cong U(\mathfrak{g}_0) = U(\mathfrak{h}) \).

- Let \( \vartheta : U(\mathfrak{p}) \longrightarrow U(\mathfrak{h}) \) be the natural projection.

**Theorem.** (P–S) The restriction of \( \vartheta \) on \( W_\chi \) is injective.

**Corollary.** The center of \( W_\chi \) coincides with the center of \( U(\mathcal{Q}(n)) \).
14. **Super-Yangian of** $Q(n)$

- Super-Yangian $Y(Q(n))$ was studied by M. Nazarov and A. Sergeev.

- $Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators

  $$T_{i,j}^{(r)} \text{ where } r = 1, 2, \ldots \text{ and } i, j = \pm 1, \pm 2, \ldots, \pm n$$

The $\mathbb{Z}_2$-grading of the algebra $Y(Q(n))$:

$$p(T_{i,j}^{(r)}) = p(i) + p(j), \text{ where } p(i) = 0 \text{ if } i > 0 \text{ and } p(i) = 1 \text{ if } i < 0$$
• We employ the formal series:

\[ T_{i,j}(u) = \sum_{r \geq 0} T_{i,j}^{(r)} u^{-r} \in Y(Q(n))[u^{-1}]. \]

• The relations in \( Y(Q(n))[u^{-1}, v^{-1}] \):

\[
(u^2 - v^2)[T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} \\
= (u + v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)) \\
- (u - v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)}
\]

• We also have the relations

\[ T_{i,j}(-u) = T_{-i,-j}(u) \]
The Main Theorem. \((P–S)\)
There exists a surjective homomorphism:

\[ \varphi : Y(Q(1)) \longrightarrow W_\chi \]

Proof.

\[ Q(1) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\} \mid a, b \in \mathbb{C} \]

\(Y(Q(1))\) is generated by \(T^{(r)}_{1,1}\) \((even)\) and \(T^{(r)}_{-1,1}\) \((odd)\) for \(r = 1, 2, \ldots\)

- **The opposite comultiplication** on \(Y(Q(1))\):

  \[ \Delta^\text{op}(T^{(r)}_{i,j}) = \sum_{s=0}^{r} \sum_{k} T^{(r-s)}_{k,j} \otimes T^{(s)}_{i,k} \]

- Homomorphism of associative algebras:

  \[ \Delta^\text{op}_n : Y(Q(1)) \rightarrow Y(Q(1)) \otimes^n \]

  \[ \Delta^\text{op}_n := \Delta^\text{op}_{n-1,n} \circ \cdots \circ \Delta^\text{op}_{2,3} \circ \Delta^\text{op} \]
• The homomorphism \( U : Y(Q(1)) \to U(Q(1)) \) defined by Nazarov and Sergeev:

\[
T_{1,1}^{(r)} \mapsto (-1)^r e_{1,1}^{(r)}, \quad T_{-1,1}^{(r)} \mapsto (-1)^r f_{1,1}^{(r)}.
\]

• Obtain the homomorphism

\[
(U \otimes_n \circ \Delta_n^{op}) : Y(Q(1)) \to U(Q(1)) \otimes_n = U(\mathfrak{h}).
\]

We have that

\[
(U \otimes_n \circ \Delta_n^{op})(T_{1,1}^{(k)}) = (-1)^k \vartheta(\pi(e_{n,1}^{(n+k-1)})) \quad (even)
\]

\[
(U \otimes_n \circ \Delta_n^{op})(T_{-1,1}^{(k)}) = (-1)^k \vartheta(\pi(f_{n,1}^{(n+k-1)})) \quad (odd)
\]

• The Harish-Chandra homomorphism \( \vartheta : W_\chi \to U(\mathfrak{h}) \) is injective. Hence

\[
\varphi := \vartheta^{-1} \circ U \otimes_n \circ \Delta_n^{op}
\]

is a surjective homomorphism \( \varphi : Y(Q(1)) \to W_\chi \).
15. Toward Non-Regular Case

- $Y(Q(n))$ is a Hopf superalgebra with **comultiplication** given by
  \[
  \Delta(T^{(r)}_{i,j}) = \sum_{s=0}^{r} \sum_{k} (-1)^{(p(i)+p(k))(p(j)+p(k))} T^{(s)}_{i,k} \otimes T^{(r-s)}_{k,j}.
  \]

- The **evaluation homomorphism** $ev : Y(Q(n)) \to U(Q(n))$ is defined by
  \[
  \left\{
  \begin{array}{ll}
  ev(T^{(1)}_{i,j}) = -e_{j,i}, & ev(T^{(1)}_{-i,j}) = -f_{j,i}, \text{ for } i, j > 0 \\
  ev(T^{(r)}_{i,j}) = 0 \text{ if } r > 1
  \end{array}
  \right.
  \]

The map
\[
\Delta_l : Y(Q(n)) \longrightarrow Y(Q(n))^\otimes l,
\]
where
\[
\Delta_l := \Delta_{l-1,l} \circ \cdots \circ \Delta_{2,3} \circ \Delta
\]
is a homomorphism of associative algebras.

Conjecture. Let $e$ be an even nilpotent element in $Q(n)$ whose Jordan blocks are all of the same size $l$. Then the finite $W$-algebra for $Q(n)$ is isomorphic to the image of $Y(Q(\frac{n}{l}))$ under the homomorphism
\[
ev^\otimes l \circ \Delta_l : Y(Q(\frac{n}{l})) \longrightarrow (U(Q(\frac{n}{l})))^\otimes l.
\]
Remark. We proved that in the case when $l = n$ (i.e. $e$ is regular),
the finite $W$-algebra for $Q(n)$ is isomorphic to the image of $Y(Q(1))$ under $U \otimes n \circ \Delta_n^{op}$.
We can now prove that

$$(U \otimes n \circ \Delta_n^{op})(Y(Q(1))) = (ev \otimes n \circ \Delta_n)(Y(Q(1))).$$
Idea of Proof. Combine all the series

\[ T_{i,j}(u) = \delta_{i,j} \cdot 1 + T_{i,j}^{(1)}u^{-1} + T_{i,j}^{(2)}u^{-2} + \ldots \]

into the single element

\[ T(u) = \sum_{i,j} E_{i,j} \otimes T_{i,j}(u) \]

of the algebra \( \text{End}(\mathbb{C}^{n|n}) \otimes Y(Q(n))[u^{-1}] \).

- The element \( T(u) \) is invertible, and we put

\[ T(u)^{-1} = \sum_{i,j} E_{i,j} \otimes \tilde{T}_{i,j}(u). \]

- The assignment \( T_{i,j}(u) \mapsto \tilde{T}_{i,j}(u) \) determines the antipodal map

\[ S : Y(Q(n)) \longrightarrow Y(Q(n)) \]

defined by M. Nazarov. It is an anti-automorphism of \( Y(Q(n)) \).
Let $n = 1$. Consider $S : Y(Q(1)) \longrightarrow Y(Q(1))$.

- The evaluation anti-homomorphism $\bar{ev} : Y(Q(1)) \rightarrow U(Q(1))$ is defined by
  
  \[
  \bar{ev}(T_{1,1}^{(1)}) = e_{1,1}, \quad \bar{ev}(T_{-1,1}^{(1)}) = f_{1,1}, \quad \bar{ev}(T_{i,j}^{(r)}) = 0 \text{ if } r > 1.
  \]

  Then
  
  \[
  \bar{ev} \circ S = U,
  \]
  
  \[
  \bar{ev} \otimes^n \circ \Delta_n \circ S = U \otimes^n \circ \Delta_n^{op}.
  \]

**Problem:** To describe the finite $W$-algebra for $Q(n)$ when $e$ is an **arbitrary** even nilpotent element.
16. Representations of $W_{\chi}$ in Regular Case

**Theorem.** (P–S) Let $M$ be a simple $W_{\chi}$-module. Then

$$\dim M \leq 2^{k+1}, \text{ where } k = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The proof is based on the Amitsur–Levitzki theorem.

**Theorem.** (A–L) If $A_1, \ldots, A_{2n}$ are $n \times n$ matrices, then

$$\sum_{\sigma \in S_{2n}} sgn(\sigma)A_{\sigma(1)} \ldots A_{\sigma(2n)} = 0.$$ 

**Idea of Proof.** The Harish-Chandra homomorphism

$$\vartheta : W_{\chi} \longrightarrow U(\mathfrak{h})$$

is injective.

$$\mathfrak{h} := \mathfrak{g}_0 = \langle e_{i,i} \ | \ f_{i,i} \rangle, \quad [f_{i,i}, f_{i,i}] = 2e_{i,i}$$
(1) \( U(\mathfrak{h}) \) satisfies A–L identity, i.e. for any \( u_1, \ldots, u_{2^k+1} \in U(\mathfrak{h}) \)

\[
\sum_{\sigma \in S_{2^k+1}} \text{sgn}(\sigma)u_{\sigma(1)} \cdots u_{\sigma(2^k+1)} = 0. \quad (*)
\]

(2) \( W_\chi \) satisfies A–L identity, since \( W_\chi \cong \vartheta(W_\chi) \subset U(\mathfrak{h}) \).

(3) Consider \( M \) as a module over the associative algebra \( W_\chi \), forgetting the \( \mathbb{Z}_2 \)-grading. Then either \( M \) is simple or \( M \) is a direct sum of two non-homogeneous simple submodules \( M_1 \oplus M_2 \).

(a) In the former case \( \dim M \leq 2^k \).

Assume \( \dim M > 2^k \). Let \( V \) be a subspace of dimension \( 2^k + 1 \). By density theorem for any \( X_1, \ldots, X_{2^k+1} \in \text{End}_\mathbb{C}(V) \) one can find \( u_1, \ldots, u_{2^k+1} \) in \( W_\chi \) such that \( (u_i)|_V = X_i \) for all \( i = 1, \ldots, 2^k+1 \). Since \( \text{End}_\mathbb{C}(V) \) does not satisfy \((*)\) we obtain a contradiction.

(b) In the latter case, we can prove in the same way that \( \dim M_1 \leq 2^k \) and \( \dim M_2 \leq 2^k \). Therefore \( \dim M \leq 2^{k+1} \).
Theorem. (P–S)
For a basic Lie superalgebra $g$, if $e$ is regular nilpotent, then all irreducible representations of $W_{\chi}$ are finite-dimensional:

$$\dim M \leq 2^{k+1}$$

$k = d$ or $k = d + 1$, where $d$ is the defect of $g$.

- $k = d$, if $g$ is of type I: $g = \mathfrak{sl}(m|n), \mathfrak{osp}(2|2n)$,
  
  or $g$ is of type II and $\dim(g^e_1)$ is even: $g = \mathfrak{osp}(2m + 1|2n)$ for $m \geq n$, 
  $\mathfrak{osp}(2m|2n)$ for $m \leq n$, $G_3$.

- $k = d + 1$, if $g$ is of type II and $\dim(g^e_1)$ is odd:
  
  $g = \mathfrak{osp}(2m + 1|2n)$ for $m < n$, $\mathfrak{osp}(2m|2n)$ for $m > n$, $D(2, 1; \alpha), F_4$. 
Idea of Proof.

1) If the good $\mathbb{Z}$-grading of $\mathfrak{g}$ with respect to $\chi$ is **even**, then there is an injective homomorphism

$$\vartheta : W_\chi \longrightarrow U(\mathfrak{g}_0).$$

2) If no good $\mathbb{Z}$-grading is **even**, then there is an injective homomorphism

$$\vartheta : W_\chi \longrightarrow \bar{W}_\chi^s,$$

where $\bar{W}_\chi^s$ is “the finite $W$-algebra” of $s$:

$s$ is the Levi subalgebra of a parabolic subalgebra $\mathfrak{p}$, such that $\mathfrak{n}^- \subset \mathfrak{m} \subset \mathfrak{p}^-$, where $\mathfrak{n}^-$ is the nilradical of the opposite parabolic $\mathfrak{p}^-$. 

$$\bar{W}_\chi^s = \left( U(s) \otimes_{U(m^s)} C_\chi \right)^{m^s},$$

where $m^s = m \cap s$, $\chi$ is the restriction of $\chi$ on $s$.

3) One can show that if $e$ is regular, then $U(\mathfrak{g}_0)$ (correspondingly, $\bar{W}_\chi^s$) satisfies Amitsur–Levitzki identity. Hence $W_\chi$ satisfies Amitsur–Levitzki identity.
17. References


